Skorohod Equation and Reflected Backward Stochastic Differential Equations

Mingyu Xu

Institute of Applied Mathematics,
Academy of Mathematics and Systems Science, CAS.

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Outline

Pricing European option

We consider a financial market, which contains one locally riskless asset S_t^0 (bond) governed by $dS_t^0 = S_t^0 r_t dt$, and n risky securities (stock) S^i is modeled by

$$dS_t^i = S_t^i [b_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dB_t^j],$$

- r is predictable, bounded and generally non-negative.
- $b = (b^1, \ldots, b^n)$ is a predictable and bounded process.
- The volatility matrix $\sigma = (\sigma^{i,j})$ is a predictable and bounded process and the inverse matrix σ^{-1} is a bounded process.
- There exists a predictable and bounded-valued process vector θ, called a risk premium, such that

$$b_t - r_t 1 = \sigma_t \theta_t, \qquad d\mathbf{P} \times dt - \text{a.s.}.$$

Let us consider a small investor whose wealth is V_t. His decision (π_t) is only based on the current information $(\mathcal{F}_t)$, i.e. $\pi = (\pi^1, \pi^2, \ldots, \pi^n)^*$ and $\pi^0 = V - \sum_{i=1}^n \pi^i$ are predictable. We say a strategy is self-financing if $V = \sum_{i=0}^n \pi^i$ satisfies

$$\begin{aligned} V_t &= V_0 + \int_0^t \sum_{i=0}^n \pi_t^i \frac{dS_t^i}{S_t^i} \\ \text{or} \quad dV_t &= r_t V_t dt + \pi_t^* \sigma_t [dB_t + \theta_t dt], \end{aligned}$$

with $\int_0^T |\sigma_t^* \pi_t|^2 \, dt < +\infty$.

Proposition 1.

Let $\xi \geq 0$ be a positive contingent claim, and in $\mathbf{L}^2(\mathcal{F}_T)$. There exists a hedging strategy (Y, π) against ξ,

$$dY_t = r_t Y_t dt + \pi_t^* \sigma_t \theta_t dt + \pi_t^* \sigma_t dB_t, Y_T = \xi,$$

and Y_t is the fair price of the contingent claim.

Outline

Backward stochastic differential equation

Backward stochastic differential equations (BSDEs in short) were first introduced by Bisumt (1973) to study stochastic maximal principle. He considered the linear case and a special non-linear case. General non-linear were first considered by Pardoux and Peng (1990).

The solution of a BSDE is a couple of progressively measurable processes (Y, Z), which satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{1}$$

where B is a Brownian motion. When terminal condition ξ is a square integrable random variable, and coefficient g satisfies Lipschitz condition and some integrable condition, BSDE (1) admits the unique solution.

Proposition 2.

Set $g(t,y,z) = r_s y + \theta_s z + a_s$. Then the solution Y of BSDE(ξ, g) is

$$Y_t = X_t^{-1} E[\xi X_T + \int_t^T a_s X_s ds | \mathcal{F}_t],$$

where $X_t = \exp(\int_0^t (r_s - \frac{1}{2}\theta_s^2) ds + \int_0^t \theta_s dB_s)$.

Theorem 2.

Let (Y^1, Z^1) (resp. (Y^2, Z^2)) be the solution of the BSDE associated with (ξ^1, g^1) (resp. BSDE(ξ^2, g^2)). Assume in addition the following: $\forall t \in [0,T]$,

$$\xi^1 \leq \xi^2, \quad g^1(t, Y_t^1, Z_t^1) \leq g^2(t, Y_t^1, Z_t^1).$$

Then $Y_t^1 \leq Y_t^2$, pour $t \in [0,T]$.

Reflected BSDEs with one barrier

In 1997, El Karoui, Kapoudjian, Pardoux, Peng and Quenez firstly published the paper with the notation of a solution of **reflected backward stochastic differential equations**(reflected BSDE in short) with a continuous barrier.
A solution for such equation associated with (ξ, f, S_t), is a triple $(Y_t, Z_t, K_t)_{0\leq t\leq T}$, which satisfies

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \qquad (2)$$

and $Y_t \geq S_t$ a.s. for any $t \leq T$, B_t is a Brownian motion. (K_t) is non decreasing continuous whose role is to push upward the process Y, in order to keep it above L. And it satisfies

$$\int_0^T (Y_s - S_s)dK_s = 0, \text{ \textit{Skorokhod condition.}} \qquad (3)$$

Optimal stopping problem and Picard iteration

Proposition [EKPPQ]

Let (Y, Z, K) be the solution of RBSDE, then

$$Y_t = ess \sup_{\tau \in \mathcal{T}_t} E[\int_t^\tau f(s, Y_s, Z_s)ds + S_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} | \mathcal{F}_t]$$

where $\mathcal{T}_t$ is be the set of all stopping times valued in $[t, T]$.

◇ Existence of solution by *Picard-type iterative procedure*,
Then prove that it is a strict contraction in an appropriated space.

Penalization method [EKPPQ]

Consider a penalized BSDE (Y^n, Z^n) with $n\int_0^t (Y_s^n - S_s)^- ds$

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + n\int_t^T (Y_s^n - S_s)^- ds - \int_t^T Z_s^n dB_s.$$

Set $K_t^n = n\int_0^t (Y_s^n - S_s)^- ds$. As $n \to \infty$, the limit of $Y^n \nearrow Y$ with $\sup_{0\leq t\leq T} |Y_t|^2 < \infty$.

Key Point: by Dini's theorem

$E(\sup_{0\leq t\leq T} |(Y_t^n - S_t)^-|^2) \to 0$, as $n \to \infty$.

With this lemma, we get

$$(Y^n, Z^n, K^n) \to (Y, Z, K) \text{ in } \mathbf{S}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \times \mathbf{S}^2_{\mathcal{F}}(0,T).$$

And the limit is the solution of reflected BSDE.

Comparison theorems for RBSDE with one barriers

Theorem 3. [General case for RBSDE's]

Let (Y^1, Z^1, K^1) (resp. (Y^2, Z^2, K^2)) be the solution of the RBSDE(ξ^1, f^1, S^1) (resp. RBSDE(ξ^2, f^2, S^2)). Assume in addition the following: $\forall t \in [0, T]$,

$$\xi^1 \leq \xi^2, \quad f^1(t, Y_t^1, Z_t^1) \leq f^2(t, Y_t^1, Z_t^1), \quad S_t^1 \leq S_t^2.$$

Then $Y_t^1 \leq Y_t^2$, pour $t \in [0, T]$.

Theorem 4. [For the comparison of K]

Set (Y^i, Z^i, K^i) $(i = 1, 2)$ to be solution of the RBSDE(ξ^i, f^i, L). If we have,

$$\xi^1 \leq \xi^2, \quad f^1(t, y, z) \leq f^2(t, y, z),$$

Then for $0 \leq s \leq t \leq T$, $Y_t^1 \leq Y_t^2$, $K_t^1 - K_s^1 \geq K_t^2 - K_s^2$.

Application: American option ([El Karoui et al.1997b])

Consider the problem of pricing an American contingent claim with payoff

$$\widetilde{S}_s = \xi 1_{\{s=T\}} + S_s 1_{\{s<T\}}.$$

Fix $t \in [0,T], \tau \in \mathcal{T}_t$; then there exists a unique strategy $(X_s(\tau, \widetilde{S}_\tau), \pi(\tau, \widetilde{S}_\tau))$, which replicate $\widetilde{S}_\tau$, i.e. for some coefficient b

$$\begin{aligned} -dX_s^\tau &= b(s, X_s^\tau, \pi_s^\tau)ds - (\pi_s^\tau)^* dB_s, 0 \le s \le T, \\ X_\tau^\tau &= \widetilde{S}_\tau. \end{aligned} \tag{4}$$

Then the price of the American contingent claim $(\widetilde{S}_s, 0 \le s \le T)$ at time t is given by

$$X_t = ess \sup_{\tau \in \mathcal{T}_t} X_t(\tau, \widetilde{S}_\tau).$$

Applying the previous results on reflected BSDE's, it follows that the price $(X_t, 0 \leq t \leq T)$ corresponds to the unique solution of the reflected BSDE associated with $(\xi,\ b,\ S)$, i.e. there exists $(\pi_t) \in \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ and $(K_t) \in \mathbf{A}^2_{\mathcal{F}}(0,T)$, such that

$$\begin{aligned} -dX_t &= b(s, X_t, \pi_t)ds + dK_t - \pi_t^* dB_t, \\ X_T &= \xi, \\ X_t \geq S_t \quad &, \quad 0 \leq t \leq T, \int_0^T (X_t - S_t)dK_t = 0. \end{aligned} \tag{5}$$

Furthermore, the stopping time $D_t = \inf(t \leq s \leq T \mid X_s = S_s) \wedge T$ is optimal, that is

$$X_t = X_t(D_t, \widetilde{S}_{D_t}).$$

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Variant Reflected BSDE

Recently a new type of reflected BSDEs has been introduced by Bank and El Karoui by a variation of Skorohod's obstacle problem, which is named as variant reflected BSDE, and has been generalized by Ma and Wang. The formulation of such equation with an optional process X (as an upper barrier)

$$Y_t = X_T + \int_t^T f(s, Y_s, Z_s, A_s)ds - \int_t^T Z_s dB_s \text{ and } Y \leq X,$$

where A is an increasing process, with $A_{0-} = -\infty$, and the flat-off condition holds $\int_t^T |Y_s - X_s| dA_s = 0$. f is decreasing in A.
In [Ma and Wang], it has been proved that the solution in a small-time duration, exists and is unique, under some conditions for f, X and ξ.

Outline

Local time and reflected local time

The constrain $Y_t \geq S_t$ implies that $\xi - S_T$ must be non-negative, and the Skorhod condition is equivalent to
$\int_0^t 1_{\{Y_s - S_s = 0\}} dK_s = K_t$, for $0 \leq t \leq T$.
Since

$$Y_0 = \xi + \int_0^T f(s, Y_s, Z_s)ds + K_T - \int_0^T Z_s dB_s$$

so that

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)ds - K_t + \int_0^t Z_s dB_s$$

and therefore the martingale part of Y is $M_t = \int_0^t Z_s dB_s$.

From Tanaka's formula

If X is a continuous semimartingale, then L^X denotes the local time of the continuous semimartingale $X - S$ at zero.

Proposition 1

Assume that $Y \geq S$ are two continuous semimartingale,

$$Y_t = Y_0 - \int_0^t f_s ds - K_t + \int_0^t Z_s dB_s \tag{6}$$

and $S = N + A$ (N is the martingale part of S and A is its variation part), where $(f_t)_{t\in[0,T]}$ is optional and $\mathbb{E}\int_0^T f_s^2 ds < \infty$, such that $\int_0^t 1_{\{Y_s=S_s\}} dK_s = K_t$. Then

$$\begin{aligned} K_t = -\int_0^t 1_{\{Y_s=S_s\}} f_s ds \quad &- \quad \int_0^t 1_{\{Y_s=S_s\}} dA_s - L_t^Y \\ 1_{\{Y_t=S_t\}}(Z_t - \sigma_t) \quad &= \quad 0. \end{aligned}$$

Skorohod's equation

Theorem (The Skorohod equation (1961))

Let $z \geq 0$ be a given number and $\varphi(\cdot) = \{\varphi(t); 0 \leq t < \infty\}$ a continuous function with $\varphi(0) = 0$. There exists a unique continuous function $l(\cdot) = \{l(t); 0 \leq t < \infty\}$, such that
(i) $x(t) := z + \varphi(t) + l(t) \geq 0; 0 \leq t < \infty$,
(ii) $l(0) = 0$, $l(\cdot)$ is nondecreasing, and
(iii) $l(\cdot)$ is flat-off $\{t \geq 0; x(t) = 0\}$; i.e. $\int_0^\infty 1_{\{x(s)>0\}} dl(s) = 0$.
This function is given by

$$l(t) = \max[0, \max_{0 \leq s \leq t}\{-(z + \varphi(s))\}], 0 \leq t < \infty.$$

Apply to Reflected BSDE

For $Y \geq S$, set $y_t = Y_{T-t} - S_{T-t}$, $L_t = K_T - K_{T-t}$ and

$$x_t = \int_{T-t}^{T} f_s ds - \int_{T-t}^{T} Z_s dB_s + S_T - S_{T-t}.$$

Then $L_0 = 0$, $t \to L_t$ increases only on $\{t : y_t = 0\}$, $y_t \geq 0$, $\eta = Y_T - S_T \geq 0$, $x_0 = 0$, and

$$y_t = \eta + x_t + L_t \ .$$

According to Skorohod's equation,

$$L_t = \max\left[0, \max_{0 \leq s \leq t} \{-(\eta + x_s)\}\right], \quad \forall t \geq 0.$$

That is for $0 \leq t \leq T$

$$L_t = \max\left[0, \max_{T-t \leq s \leq T}\left\{-\left(Y_T + \int_s^T f_r dr - S_s - \int_s^T Z_r dB_r\right)\right\}\right].$$

We may recover $K_t = L_T - L_{T-t}$ to obtain

$$\begin{aligned} K_t &= \max\left[0, \max_{0 \leq s \leq T}\left\{-\left(Y_T + \int_s^T f_r - S_s - \int_s^T Z_r dB_r\right)\right\}\right] \\ &\quad - \max\left[0, \max_{t \leq s \leq T}\left\{-\left(Y_T + \int_s^T f_r dr - S_s - \int_s^T Z_r dB_r\right)\right\}\right]. \end{aligned}$$

Reflected BSDE with resistance (joint work with Zhongmin Qian)

We study the following stochastic integral equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, K_s)ds + K_T - K_t - \int_t^T Z_s dB_s \qquad (7)$$

for $t \leq T$, subject to the constrain that

$$Y_t \geq S_t \text{ and } \int_0^T (Y_t - S_t)dK_t = 0, \qquad (8)$$

S is a continuous semimartingale such that $\sup_{t\leq T} S_t^+$ is square integrable, and $\xi \in \mathcal{L}^2(\mathcal{F}_T)$, which are given data.

Assumption for f

$|f(s,y,z,k) - f(s,y',z',k')| \leq C_1(|y-y'| + |z-z'|) + C_2|k-k'|$ where C_1 and C_2 are two constants, and $\mathbb{E}\int_0^T f^0(t)^2 dt < \infty$, with $f^0(t) \equiv f(t,0,0,0)$.

Definition

By a solution triple (Y, Z, K) of the terminal problem (7) we mean that $Y \in \mathcal{S}^2(0,T)$, $K \in \mathcal{A}^2(0,T)$ and K is optional, and $Z \in \mathcal{H}_d^2(0,T)$, which satisfies the stochastic integral equations (7) with time t running from 0 to T.

The integral equation (7) is not local in time, since K will be path dependent over the whole range $[0,T]$. This is the reason why we have to require the Lipschitz constant C_2 in Assumption for f to be small.
If (Y, Z, K) is a solution of (7)-(8), then we must have

$$\begin{aligned} K_t &= \max\left[0, \max_{0\le s\le T}\left\{-\left(\xi - S_s + \int_s^T f(r, Y_r, Z_r, K_r)dr - \int_s^T Z_r dB_r\right)\right\}\right. \\ &\quad - \max\left[0, \max_{t\le s\le T}\left\{-\left(\xi - S_s + \int_s^T f(r, Y_r, Z_r, K_r)dr - \int_s^T Z_r dB_r\right)\right.\right. \end{aligned}$$

Outline

Constructing Picard's iteration

We construct

$$\begin{array}{ccc} \mathcal{S}^2(0,T)\times\mathcal{H}_d^2(0,T)\times\mathcal{A}^2(0,T) & \to & \mathcal{S}^2(0,T)\times\mathcal{H}_d^2(0,T)\times\mathcal{A}^2(0,T) \\ (Y,Z,K) & \to & (\tilde{Y},\tilde{Z},\tilde{K}) \end{array}$$

Here $\tilde{Z}.B$ is the martingale part of $\tilde{Y}$. We first define

$$\begin{aligned} \tilde{K}_t &= \max\left[0, \max_{0\le s\le T}\left\{-\left(\xi+\int_s^T f(r,Y_r,Z_r,K_r^\flat)dr - S_s - \int_s^T Z_r dB_r\right)\right\}\right. \\ &\quad -\max\left[0, \max_{t\le s\le T}\left\{-\left(\xi+\int_s^T f(r,Y_r,Z_r,K_r^\flat)dr - S_s - \int_s^T Z_r dB_r\right)\right.\right. \end{aligned}$$

where $K_r^\flat$ is the optional projection of K, as we do not assume that K is optional, but we want to ensure that the arguments in the driver f are optional.

We are going to define $\tilde{M}$ and $\tilde{Y}$. The natural way is

$$\hat{Y}_t = \xi + \int_t^T f(s, Y_s, Z_s, K_s^\flat)ds + \tilde{K}_T - \tilde{K}_t - \int_t^T Z_s dB_s. \quad (10)$$

$\hat{Y}$ is however not necessary adapted. Therefore we define $\tilde{Y}$ to be its optional projection $\hat{Y}^\flat$:

$$\begin{aligned} \tilde{Y}_t &= \mathbb{E}\left\{ \xi + \int_t^T f(s, Y_s, Z_s, K_s^\flat)ds + \tilde{K}_T - \tilde{K}_t - \int_t^T Z_s dB_s \middle| \mathcal{F}_t \right\} \\ &= \mathbb{E}\left\{ \xi + \int_t^T f(s, Y_s, Z_s, K_s^\flat)ds + \tilde{K}_T - \tilde{K}_t \middle| \mathcal{F}_t \right\}. \quad (11) \end{aligned}$$

According to Skorohod's equation, $\hat{Y} \geq S$, so is $\tilde{Y}$.
Moreover $\tilde{K}$ increases only on $\{t : \hat{Y}_t - S_t = 0\}$, which however does not necessarily coincide with the level set $\{t : \tilde{Y}_t - S_t = 0\}$.

Notice $\tilde{N}_t = \tilde{K}^\flat_t - \tilde{K}^o_t$ is a continuous martingale. Therefore the martingale part of $\tilde{Y}$ is

$$\tilde{M}_t = \mathbb{E}\left\{ \xi + \tilde{K}_T + \int_0^T f(s, Y_s, Z_s, K^\flat_s) ds \middle| \mathcal{F}_t \right\} - \tilde{N}_t$$

So we define the density predictable process $\tilde{Z}$ by Itô's martingale representation $\tilde{M}_t - \tilde{M}_0 = \int_0^t \tilde{Z}_s . dB_s$, so that

$$\tilde{Y}_t = \xi + \int_t^T f(s, Y_s, Z_s, K^\flat_s) ds + \tilde{K}^o_T - \tilde{K}^o_t - \int_t^T \tilde{Z}_s . dB_s.$$

The mapping $\mathfrak{L} : (Y, Z, K) \to (\tilde{Y}, \tilde{Z}, \tilde{K})$ is thus well defined.

Proposition 1.

If (Y, Z, K) is a fixed point of $\mathfrak{L}$, then (Y, Z, K) is a solution the reflected BSDE (7)-(8).

Proof. Suppose (Y, Z, K) is a fixed point of the non-linear mapping $\mathfrak{L}$, so that

$$M_t = \mathbb{E}\left\{\xi + \int_0^T f(s, Y_s, Z_s, K_s^\flat)ds + K_T - K_t | \mathcal{F}_t\right\} + K_t^o,$$

$$Y_t = \mathbb{E}\left\{\xi + \int_t^T f(s, Y_s, Z_s, K_s^\flat)ds + K_T - K_t | \mathcal{F}_t\right\}$$

$$\begin{aligned} K_t &= \max\left[0, \max_{0\le s\le T}\left\{-\left(\xi + \int_s^T f(r, Y_r, Z_r, K_r^\flat)dr - S_s - \int_s^T Z_r dB_r\right)\right\}\right. \\ &\quad - \max\left[0, \max_{t\le s\le T}\left\{-\left(\xi + \int_s^T f(r, Y_r, Z_r, K_r^\flat)dr - S_s - \int_s^T Z_r dB_r\right)\right.\right. \end{aligned}$$

Then $Y_T = \xi$ and

$$Y_t = M_t - K_t^o - \int_0^t f(s, Y_s, Z_s, K_r^\flat) ds,$$

so that

$$\xi - Y_t = \int_t^T Z_s dB_s - (K_T^o - K_t^o) - \int_t^T f(s, Y_s, Z_s, K_r^\flat) ds \ .$$

According to the uniqueness of the Skorohod's equation, it follows that $K^o = K$. therefore K is adapted, and $K = K^\flat = K^o$. That completes the proof.

Outline

Estimates

Let us prove the following key a priori estimate about $\mathfrak{L}$. Let $(\tilde{Y}, \tilde{Z}, \tilde{K}) = \mathfrak{L}(Y, Z, K)$ and $(\tilde{Y}', \tilde{Z}', \tilde{K}') = \mathfrak{L}(Y', Z', K')$. Let $D_t = e^{\alpha t}|Y_t - Y_t'|^2$ and $\tilde{D}_t = e^{\alpha t}|\tilde{Y}_t - \tilde{Y}_t'|^2$.

Proposition 2.

Suppose f satisfies the assumption of f. Then for any $\alpha \geq 0$, $\varepsilon > 0$ and $\varepsilon' > 0$ we have

$$\begin{aligned} \mathbb{E}\left(\tilde{D}_0\right) \leq\ & -(\alpha - \varepsilon C_1 - \varepsilon' C_2)||\tilde{Y} - \tilde{Y}'||_\alpha^2 - ||\tilde{Z} - \tilde{Z}'||_\alpha^2 \\ & + \frac{2C_1}{\varepsilon}\left(||Y - Y'||_\alpha^2 + ||Z - Z'||_\alpha^2\right) \\ & + \frac{2C_2}{\varepsilon'}||K^\flat - K'^\flat||_\alpha^2 \end{aligned} \tag{12}$$

where $||\tilde{Y} - \tilde{Y}'||_\alpha^2 = \int_0^T e^{\alpha t}|\tilde{Y}_t - \tilde{Y}_t'|^2 dt$.

Proof. By Itô formulae, and the fact that for an optional process φ

$$\mathbb{E}\int_t^T \varphi_s d\left(\tilde{K}_s^o - \tilde{K}_s'^o\right) = \mathbb{E}\int_t^T \varphi_s d\left(\tilde{K}_s - \tilde{K}_s'\right)$$

taking expectation to obtain

$$\begin{aligned}
\mathbb{E}\tilde{D}_t &= -\alpha\int_t^T \mathbb{E}\left(\tilde{D}_s\right) ds - \mathbb{E}\int_t^T e^{\alpha s} d\langle\tilde{M} - \tilde{M}'\rangle_s \\
&\quad +2\mathbb{E}\int_t^T e^{\alpha s}\left(\tilde{Y}_s - \tilde{Y}_s'\right) d(\tilde{K}_s - \tilde{K}_s') \\
&\quad +2\int_t^T \mathbb{E}\left\{e^{\alpha s}\left(\tilde{Y}_s - \tilde{Y}_s'\right)\left[f(s, Y_s, Z_s, K_s^\flat) - f(s, Y_s', Z_s', K'^\flat_s)\right]\right\} ds,
\end{aligned}$$

There is an important observation due to [EKPPQ],

$$\begin{aligned}
&\mathbb{E}\int_t^T e^{\alpha s}\left(\tilde{Y}_s - \tilde{Y}_s'\right) d(\tilde{K}_s - \tilde{K}_s') \\
\leq\ &\mathbb{E}\int_t^T e^{\alpha s}(\tilde{Y}_s - S_s) d\tilde{K}_s + \mathbb{E}\int_t^T e^{\alpha s}(\tilde{Y}_s' - S_s) d\tilde{K}_s'.
\end{aligned}$$

Moreover, according to Skorohod's equation, $\tilde{K}$ increases only on $\{s : \hat{Y}_s - S_s = 0\}$ so that

$$\mathbb{E}\int_t^T e^{\alpha s}(\hat{Y}_s - S_s)d\tilde{K}_s = 0.$$

Since $\tilde{Y}$ is the optional projection of $\hat{Y}$, and $\tilde{K}^o$ is the dual optional projection of $\tilde{K}$, therefore

$$\begin{aligned}\mathbb{E}\int_t^T e^{\alpha s}(\tilde{Y}_s - S_s)d\tilde{K}_s &= \mathbb{E}\int_t^T e^{\alpha s}(\tilde{Y}_s - S_s)d\tilde{K}^o_s \\ &= \mathbb{E}\left(\int_t^T e^{\alpha s}(\hat{Y}_s - S_s)d\tilde{K}_s\right)^o.\end{aligned}$$

Since $\tilde{K}$ increases only on $\{s : \hat{Y}_s - S_s = 0\}$, so that $\int_t^T e^{\alpha s}(\hat{Y}_s - S_s)d\tilde{K}_s = 0$ and therefore $\mathbb{E}\int_t^T e^{\alpha s}(\tilde{Y}_s - S_s)d\tilde{K}_s = 0$.
Similarly $\mathbb{E}\int_t^T e^{\alpha s}(\tilde{Y}'_s - S_s)d\tilde{K}'_s = 0$.
Then result follows from Lipschitz assumption on f.

Proposition 3.

We have

$$\begin{aligned} ||\tilde{K} - \tilde{K}'||_{\infty}^2 \leq & \ \left(24TC_1^2 + 4C_3\right)\left(||Y - Y'||_0^2 + ||Z - Z'||_0^2\right) \\ & + 24T^2C_1^2||K - K'||_{\infty}^2 \end{aligned}$$

where $||K - K'||_{\infty}^2 = \sup_{0\leq t\leq T} \mathbb{E}|K_s - K'_s|^2$, where C_3 is the constant appearing in the Burkholder inequality.

Lemma

Let φ, ψ be two continuous paths in $\mathbb{R}^1$. Then

$$\left|\sup_{s\leq t} \varphi_s - \sup_{s\leq t} \psi_s\right| \leq \sup_{s\leq t} |\varphi_s - \psi_s|.$$

Existence

Theorem 1

Assume f satisfies Assumption of f, and there is a constant $C_0 > 0$ depending on C_1 and T such that if $C_2 \leq C_0$, then there is a unique solution (Y, Z, K) to the problem (7)-(8). Moreover the reversed local time satisfies (9). If $C_2 = 0$ that is the driver f does not depend on K, then there is no restriction on C_2.

Proof. Let $\alpha \geq 0$ and $\beta > 0$ to be chosen late, and define

$$||(Y,Z,K)-(Y',Z',K')||^2_{\alpha,\beta} = ||Y-Y'||^2_\alpha + ||Z-Z'||^2_\alpha + \beta||K-K'||^2_\infty.$$

Let $(\tilde{Y}, \tilde{Z}, \tilde{K}) = \mathfrak{L}(Y, Z, K)$ and $(\tilde{Y}', \tilde{Z}', \tilde{K}') = \mathfrak{L}(Y', Z', K')$.
Then

$$||K^\flat - K'^\flat||^2_\alpha \leq \frac{e^{\alpha T} - 1}{\alpha} ||K - K'||^2_\infty$$

Then from estimation results and well chosen parameters, we get that Then there is a number $C_0 > 0$ such that if $C_2 \leq C_0$,

$$||(\tilde{Y}, \tilde{Z}, \tilde{K}) - (\tilde{Y}', \tilde{Z}', \tilde{K}')||_{\alpha,\beta} \leq \frac{1}{\sqrt{2}} ||(Y, Z, K) - (Y', Z', K')||_{\alpha,\beta},$$

Remark

Here we set C_0 to be the solution of

$$\frac{3x^2T^2}{4(3TC_1^2 + C_3)} + x\frac{e^{(1+8C_1^2+x)T} - 1}{1 + 8C_1^2 + x} = \frac{1}{64\,(3TC_1^2 + C_3)},$$

which is a candidate of the boundary of Lipschitz constant of K.

Remark

Similarly, we can change the assumption by: there is a constant C_0 depending on C_1 and C_2 such that if $T \leq C_0$, then the existence of the solution holds.

Outline

Continuous dependence and uniqueness

Proposition 4: A priori estimate

Under the same assumptions in Theorem 1. Suppose (Y, Z, K) to be the solution of reflected BSDE(7), then there exists a constant C depending only on C_1, C_2 and T, such that

$$\mathbb{E}\left(\sup_{0\le t\le T} Y_t^2 + \int_0^T |Z_s|^2 ds + K_T^2\right) \le C\mathbb{E}\left(\xi^2 + \int_0^T (f_t^0)^2 dt + (\sup_{0\le t\le T} S_t^+)^2\right).$$

Remark

Here we may choose C_0 such that $C_4 C_2^2 T^2 + 4C_2^2 T \le \frac{1}{2}$ and set $\alpha = 4C_4$, then the result holds. It is one candidate for the estimation. Meanwhile we can also replace the boundary condition of C_2, by the boundary condition of T, as before.

Theorem 2.

Under the same assumptions in Theorem 1. Suppose (Y^i, Z^i, K^i), $(i = 1, 2)$ to be the solution of reflected BSDE (7) with parameters (ξ^i, f^i, S^i), respectively. Set

$$\begin{aligned} \triangle Y &= Y^1 - Y^2,\ \triangle Z = Z^1 - Z^2,\ \triangle K = K^1 - K^2, \\ \triangle \xi &= \xi^1 - \xi^2,\ \triangle f = f^1 - f^2,\ \triangle S = S^1 - S^2. \end{aligned}$$

Then

$$\begin{aligned} & \mathbb{E}\left(\sup_{0\le t\le T} |\triangle Y_t|^2 + \int_0^T |\triangle Z_s|^2 ds + \sup_{0\le t\le T} |\triangle K_t| \right) \\ \le\ & C\mathbb{E}\left(\triangle\xi^2 + \int_0^T \left|\triangle f(t, Y_t^1, Z_t^1, K_t^1)\right|^2 dt \right) \\ & + C\Psi^{\frac{1}{2}}_{\xi^1,\xi^2,f^1(0),f^2(0),S^1,S^2,T} \left[\mathbb{E}(\sup_{0\le t\le T} |\triangle S_t|)^2 \right]^{\frac{1}{2}} \end{aligned}$$

Optimal stopping representation

Proposition 3.

Let (Y, Z, K) be the solution of reflected BSDE with resistance, then

$$Y_t = ess \sup_{\tau \in \mathcal{T}_t} E[\int_t^\tau f(s, Y_s, Z_s, K_s)ds + S_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_t]$$

where $\mathcal{T}_t$ is be the set of all stopping times valued in $[t, T]$.

The prove is same as in paper [EKPPQ].

Comparison Theorem

Consider (Y^i, Z^i, K^i), $i = 1, 2$, to satisfy

$$\begin{aligned} Y_t^i &= \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i, K_s^i)ds + K_T^i - K_t^i - \int_t^T Z_s^i dB_s, \\ Y_t^i &\geq S_t^i, \quad \int_0^T (Y_s^i - S_s^i)dK_s^i = 0. \end{aligned}$$

Assumption for comparison

$$\xi^1 \leq \xi^2, \quad f^1(t, y, z, k) \leq f^2(t, y, z, k), \quad S_t^1 \leq S_t^2.$$

Proposition 5.

If $f^1(t, y, z, k)$ is decreasing in k and $f^2(t, y, z, k)$ is increasing in k, with $f^1(t, y, z, 0) \leq f^2(t, y, z, 0)$, then $Y_t^1 \leq Y_t^2$.

Compare with classic reflected BSDE

Proposition 6.

If $f^1(t,y,z,k)$ is decreasing in k, and

$$f^1(t,y,z,0) \leq f^2(t,y,z),$$

then $Y_t^1 \leq Y_t^2$. Here (Y^2, Z^2, K^2) is the solution of reflected BSDE without resistence.

Proposition 7.

If $f^2(t,y,z,k)$ is increasing in k, and

$$f^1(t,y,z) \leq f^2(t,y,z,0),$$

then $Y_t^1 \leq Y_t^2$. Here (Y^1, Z^1, K^1) is the solution of reflected BSDE without resistence.

Outline

Skorohod equation in multi-dimensional case

Definition. (Skorohod solution)

Let $f : \mathbb{R}_+ \to \mathbb{R}^d$ be a (continuous) path with $f_0 \in \overline{D}$. A pair (g, l) is a solution to the Skorohod problem $S(f; D)$ if
(i) $g : \mathbb{R}_+ \to \bar{D}$ is a path in $\overline{D}$;
(ii) $l : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and increases only when $g_t \in \partial D$:

$$l_t = \int_0^t 1_{\partial D}(g_s) dl_s;$$

(iii) the Skorohod equation holds:

$$g_t = f_t + \int_0^t n(g_s) dl_s.$$

We have the following results (see Lions and Sznitman(84) and Theorem 2.5 and Remark 3 in Hsu thesis).

Theorem

Let D be a domain in $\mathbb{R}^d$ with C^1 boundary and f a continuous path in $\mathbb{R}^d$ such that $f_0 \in \overline{D}$. Then there exists a solution to the Skorohod problem $S(f; D)$. The solution is unique if D has a C^2 boundary. Furthermore, the moduli of continuity of f and g satisfy

$$\Delta_s(\sigma; f) \leq C\Delta_s(\sigma; f),$$

where $\Delta_s(\sigma; f) = \sup\{\|f_a - f_b\| : a, b \in [0, s+\sigma], |a-b| \leq \sigma\}$. The constant C depends on the domain D and the bounds of s and σ, but not on φ.

Remark

The inequality between the moduli of continuity of f and g says that g is no less continuous than f.

Outline

Reflected BSDE in multi-dimension on C^2 domain (joint work with Elton Hsu)

By solution of a BSDE on a bounded domain D with reflecting boundary condition we mean a triple (Y, Z, K) such that $Y_t \in \overline{D}$

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T n(Y_s)\, dK_s - \int_t^T Z_s\, dB_s.$$

And the increasing process K increases only when Y is at the boundary, i.e.,

$$K_t = \int_0^t I_{\partial D}(Y_s)\, dK_s.$$

Let $X_t = Y_{T-t}$, $L_t = K_T - K_{T-t}$, and
$F(Y,Z)_t = Y_T + \int_{T-t}^{t} f_s\,ds - \int_{T-t}^{T} Z_s\,dB_s$, then

$$X_t = F(Y,Z)_t + \int_0^t n(X_s)\,dL_s,$$

so (X,L) is the solution of the Skorohod problem $S(F(Y,Z);D)$.

Theorem

The reflected BSDE on D, which has C^2 boundary, has at most one solution $(Y,Z,K) \in\in S_d^2(0,T) \times H_{d\times n}^2(0,T) \times FV_d^2(0,T)$.

Existence.

- D is convex, existence holds by Skorohod equation and Picard iteration. ([Gegoux-Petit&Pardoux] with penalization method)
- D is non-convex, the solution may not exist.

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Thanks for your attention!

Q & A